Volume 71

2023

A pure theory of population distribution when preferences are ordinal

By ODED STARK *and* GRZEGORZ KOSIOROWSKI

Reprinted from

THE ANNALS OF REGIONAL SCIENCE

The Annals of Regional Science (2023) 71:317–342
https://doi.org/10.1007/s00168-022-01194-y

ORIGINAL PAPER

A pure theory of population distribution when preferences are ordinal

Oded Stark[1,2] · Grzegorz Kosiorowski[3]

Received: 23 January 2022 / Accepted: 3 November 2022 / Published online: 22 December 2022

Abstract

We model an environment in which individuals prefer to be in a space in which their rank is higher, be it a social space, a geographical space, a work environment, or any other comparison sphere which we refer to in this paper, and without loss of generality, as a region. When the individuals can choose between more than two regions, we inquire: (i) whether a steady-state distribution of the population is reached; (ii) how long it will take to reach a steady state; and (iii) if a steady state obtains, whether at the steady state social welfare is maximized. Despite the fact that when there are three or more regions the mobility paths are more intricate than when there are only two regions, we prove that a steady-state distribution of the population across the regions is reached; we identify the upper bound of the number of time periods that it will take to reach the steady-state distribution; and we show that the steady-state distribution maximizes social welfare. This last result is surprising: even though the individuals act of their own accord, they achieve the socially preferred outcome.

Keywords: Inter-space mobility; Three or more spaces; Ordinal preferences; Distaste for low rank; Steady-state population distribution; Social welfare

JEL Classification: C61; C62; D50; D60; D62; I31; R13; R23; Z13

We are indebted to a reviewer for sound advice and thoughtful commentary.

✉ O. Stark
ostark@uni-bonn.de

1 University of Bonn, Bonn, Germany

2 University of Warsaw, Warsaw, Poland

3 Cracow University of Economics, Cracow, Poland

"The desire of . . . obtaining rank among our equals, is, perhaps, the strongest of all our desires" (Smith, 1759, Part VI, Section I, Paragraph 4).

1. Introduction

A large number of empirical works demonstrate that when people are ranked low, they take action to improve their rank. Studies showing this are, for example, of workers (Falk and Ichino, 2006; Mas and Moretti, 2009; Bandiera et al., 2010; Cohn et al., 2014), and of students (Sacerdote, 2001; Hanushek et al., 2003; Azmat and Iriberri, 2010; Bursztyn and Jensen, 2015; Garlick, 2018; Dobrescu et al., 2021). Heffetz and Frank (2011) review the significance of social status (rank) in economic affairs. Noy and Sin (2021) document the importance of rank for happiness, whether the reference group consists of neighbors or of co-workers. Drawing on data from New Zealand, Noy and Sin find that while ordinal (rank) comparisons matter for subjective wellbeing, cardinal (absolute income) comparisons do not. (Noy and Sin provide many references to studies that establish the importance of rank in the determination of wellbeing.) It is not difficult to conceptualize the choice of a comparison sphere as a response to dislike of being ranked low. Other things being equal, people will prefer to be in a space in which their rank is higher, be it a social space, a geographical space, a work environment, or any other comparison sphere which we refer to henceforth in this paper, and without loss of generality, as a region. Relocation can take them there.

In "Migration when social preferences are ordinal: Steady-state population distribution and social welfare" Stark (2017) studies the decisions of individuals as to which of two regions to locate to when the choice is governed by ordinal social preferences, specifically by a distaste for low rank in the hierarchy of incomes. What is meant by rank is the rung occupied by an individual in the hierarchy of the incomes in the region in which the individual is located. Thus, in a given region, the individual whose income is the highest occupies the top rank, namely his rank is first, the rank of the individual whose income is the second highest is second, namely he occupies a rung that is just below the top rung, and so on.

There are three core claims of Stark's (2017) model. These claims pertain to issues of the steady-state distribution of the population between the regions, and to social welfare.

Claim 1 considers rational but not sophisticated individuals in the sense that they do not form expectations as to how other individuals will behave; the individuals optimize their own utility, but they do not take into consideration that other individuals do likewise. We refer to these individuals as "near-sighted." There are $n \geq 4$ (n is a finite natural number) individuals who are initially all located in the same region. Claim 1 establishes that a steady-state distribution is reached by time period $\frac{n}{2}$ if n is even, or by time period $\frac{n-1}{2}$ if n is odd. At the steady state, the individuals are distributed between the two regions evenly or evenly but for one.

Claim 4 in Stark's model refers to an alternative setting in which the individuals are "far-sighted:" in considering a move between regions, individuals who are "far-sighted" figure out and take into account the decisions that will be taken by other individuals who are higher up in the hierarchy of the income distribution. The claim states that in that setting, a steady-state distribution is reached in just one time period, and that at the steady state, the individuals are distributed evenly, or evenly but for one, between the two regions. In both cases, namely when the individuals are "near-sighted," and when the individuals are "far-sighted," it is shown that the steady-state distribution maximizes social welfare, where social welfare is defined as the negative of the sum of the ranks of the individuals. (For example, when two individuals with different incomes are located in the same region with ranks first and second, social welfare is (-3); if they locate in different regions so that each is ranked first, social welfare is higher at (-2).)

In Claim 5 in Stark's model it is stated that any allocation of the individuals in which they are distributed evenly (or evenly but for one) between the two regions constitutes the social optimum. This claim is intriguing because although the individuals act of their own accord,[1] they achieve the socially preferred outcome.

The aim of the current paper is to inquire whether the core claims of Stark's 2017 model are robust to settings of more than two regions. A setting of three or more regions is closer to reality than a setting of two regions, but is more difficult to handle.[2] The main outcome of the current paper's analysis is robustness of the core claims of Stark's 2017 model, that the paths leading to the steady-state distributions are harder to chart notwithstanding. To demonstrate this robustness, we proceed as follows. In Sub-section 2.1 we list our modeling assumptions and several key concepts. In Sub-section 2.2 we comment briefly, first, on a setting in which there are three or more regions and two individuals and, second, on a setting in which there are three or more regions and three individuals. In these two settings, it does not matter whether the individuals are "near-sighted" or "far-sighted:" steady-state distributions are obtained, and they are the same for both types of individuals. In Sub-section 2.3 we study the case of more than three "near-sighted" individuals, and in Sub-section 2.4 we study the case of more than three "far-sighted" individuals. In both cases we find that the individuals' behavior adds up to steady-state distributions, and we characterize both the processes that lead to the steady-state distributions, and the steady-state distributions themselves. Whereas in these two respects the two cases differ, in terms of *reaching* a steady-state distribution, the cases do not differ. In Section 3 we analyze social welfare. In Section 4 we address complementary reflections. In Section 5 we present concluding remarks.

[1] After all, by their location choices, the individuals create negative externalities in that the very nature of the dynamics of their movement is to decrease the rank-utility of every lower-ranked individual in the region to which they move.

[2] An early example of modeling location choice when there are three locations to choose from is a study by Shukla and Stark (1986).

2. An economy of three or more regions: Steady-state distributions

2.1 Assumptions and concepts

We let n individuals (numbered $1, 2, \ldots, n$) move between k regions where n and k are finite natural numbers, $n \geq 2$, and $k \geq 3$.[3] The regions are not arranged in a spatial structure, and in all relevant aspects they are identical. Thus, the results obtained in this paper are invariant to arbitrary permutation of the labels (names) of the regions. The individuals have given incomes (individual 1 with income x_1, individual 2 with income x_2, and so on, where $0 < x_1 < x_2 < \ldots < x_n$). The individuals are concerned about their rank, namely their position in the hierarchy of incomes: the rank of individual $i \in \{1, 2, \ldots, n\}$ in a region is the number of individuals in the region whose incomes are higher than the income of individual i plus one. For example, the rank of the individual with the highest income in a region is first, and if among the individuals residing in the same region as individual i there are exactly four individuals whose income exceeds his income, then the rank of individual i is 5. Improvement in rank is the sole reason for considering moving from one region to another. The individuals can move between regions at no cost. (In Section 4 we provide conditions under which the reported results are invariant to a relaxation of the no-cost assumption.) When residing in any other region is not more appealing than residing in a given region, the individuals do not move. The region in which an individual is located is the individual's exclusive sphere of comparison. Using a slightly different terminology, the region where an individual resides is the individual's "isolated" environment in which rank comparisons take place. Disutility arises when an individual's rank is any but the top one, and is proportional to the number of individuals who occupy higher ranks in the individual's region. An individual can move from one region to another as many times as he wishes. A move from one region to another region takes one time period. Because, as already noted, movement between the regions is cost free, if an individual finds that a move results in a worse outcome, he can correct for that by moving back, or by moving to a different region. Reallocation to another region implies an association with a different hierarchy of incomes, but involves no rise in income; improving rank is the sole motive. In any time period, the individuals know the incomes and the locations of all the other individuals, but they are not informed about each other's moving plans.

We consider two possible mechanisms or procedures by which individuals assess whether to move, and where to move to in any given time period. In the base model (Sub-section 2.3), the individuals are rational but not sophisticated; they do not form expectations as to how other individuals will behave, and they move to the region in which they would have the best rank if no one else moves. We refer to these individuals as "near-sighted," and to the associated dynamics as "near-sightedness." In a subsequent model (Sub-section 2.4), the "near-sightedness" assumption is replaced by the assumption of "far-sightedness:" in considering a move between regions, individuals who are "far-sighted" figure out and take into account the decisions that will be taken by other individuals who are positioned higher up

[3] The case of $k = 2$ is studied thoroughly in Stark (2017).

in the hierarchy of the income distribution. These individuals move to a region in which they can improve their rank after figuring out the anticipated movement of other individuals. The nature of the information processed by the individuals differs between the two settings. In the "near-sighted" setting the individuals respond ex-post to information that they obtain from *observing* the whereabouts of individuals who are higher up in the hierarchy of the income distribution. In the "far-sighted" setting the individuals *anticipate* the whereabouts of individuals who are higher up in the hierarchy of the income distribution; they respond to that information ex ante. In both settings the individuals' decision to move is based on the location of individuals higher up in the hierarchy, whether this location is observed or anticipated. Detailed explanations of the decision-making mechanisms are presented in the respective sub-sections.

Additionally, we assume that, to begin with, all the individuals are in the same region, which we label as A in the case of three regions, and as A_1 in the general case of multiple (three or more) regions. We make this assumption for two reasons: it happens to yield the most complex case, so once we deal with that case, the analysis of other cases follows smoothly; and drawing on this assumption simplifies the notation considerably. However, the reported results are invariant to the initial distribution of the individuals between the regions: the assumption that at the outset all the individuals reside in the same region is not necessary for obtaining our results. In Section 4 we discuss the implication of relaxing this assumption. We show that when the assumption is discarded, the only change is that the final distribution is reached faster.

2.2 Two or three individuals: The cases of "near-sighted" individuals, and of "far-sighted" individuals

We let $n = 2$ or $n = 3$ individuals be in region A. Then two or more other regions that in all relevant respects are identical to region A become available.

If $n = 2$, then a steady state is reached in just one time period (henceforth we label the stages of the process as time periods): individual 1 whose rank is second in region A moves to any of the other regions. Individual 2 who has nothing to gain from moving does not move. This result obtains regardless of whether the individuals are "near-sighted" or "far-sighted."

If $n = 3$, then in the first time period, individuals 2 and 1 move. Individual 3 who has nothing to gain from moving to another region does not move and remains in region A. When individuals 2 and 1 move, they either locate to different regions, in which case a steady-state distribution is reached with one individual located in each of three regions, or individuals 2 and 1 happen to move to the same region, say region B. In this second case, individual 1 will subsequently move to yet another region, say region C (individual 1 will not move back to region A because whereas in region C - meaning any region other than regions A and B - he will occupy the top rank, in region A he will be ranked second), at which time a steady-state distribution of the three individuals is reached, again with one individual in each of the populated regions, and with the other regions remaining uninhabited. In this second case, reaching the steady state takes longer, however: two time periods rather than one. In this $n = 3$ case, as in the $n = 2$ case, the result obtains regardless of whether the individuals are "near-sighted" or "far-sighted."

Naturally, the really interesting and quite challenging cases to study are those in which there are more individuals than regions. We examine these cases next. We separate our treatment of the "near-sighted" case from that of the "far-sighted" case because in terms of the number of time periods that it takes to reach a steady-state distribution and in terms of the steady-state distribution itself, the two cases differ.

2.3 Four or more individuals: The case of "near-sighted" individuals

Let $\{1, 2, \ldots, n\}$ be a set of $n \geq 4$ "near-sighted" individuals, and let there be $k \geq 3$ regions: $A_1, A_2, \ldots, A_k$. In proceeding, we illustrate our considerations by means of examples of $k = 3$ regions, in which case we use the notation $A_1 = A$, $A_2 = B$, and $A_3 = C$. We term $(X_1, X_2, \ldots, X_k)$ a distribution if $X_1, X_2, \ldots, X_k$ are the sets of individuals who reside, respectively, in regions $A_1, A_2, \ldots, A_k$. In particular, $X_1 \cup X_2 \cup \ldots \cup X_k = \{1, 2, \ldots, n\}$. When $i \in X$, we denote by $R(i, X)$ the rank of individual i in the set X.

Definition 1. A "near-sighted" mobility trajectory is a sequence of distributions $(T_m)_{m=0}^{\infty} = (X_1^m, X_2^m, \ldots, X_k^m)_{m=0}^{\infty}$ such that:

(i) $T_0 = (\{1, 2, \ldots, n\}, \varnothing, \ldots, \varnothing)$ namely in time period 0 all the individuals reside in region A_1.

(ii) If $i \in X_\alpha^{m+1}$ for $\alpha \in \{1, 2, \ldots, k\}$ and $m \geq 0$, then $R(i, X_\alpha^m \cup \{i\}) = \min\limits_{j \in \{1,2,\ldots,k\}} R(i, X_j^m \cup \{i\})$.

(iii) If $i \in X_\alpha^m$ for $\alpha \in \{1, 2, \ldots, k\}$, $m \geq 0$ and $R(i, X_\alpha^m) = \min\limits_{j \in \{1,2,\ldots,k\}} R(i, X_j^m \cup \{i\})$, then $i \in X_\alpha^{m+1}$.

Definition 1 lists three attributes of the dynamics of mobility. Attribute (i) states that to begin with, all the individuals are in the same region A_1. Attribute (ii) states that each individual in each time period chooses a region in which he can obtain his best possible rank (conditional on him assuming - recalling that he is "near-sighted" - that no one else moves). Attribute (iii) states that in each time period an individual who resides in the region where he already obtains his best possible rank does not move. Given the opening distribution of the individuals as per (i), a mobility trajectory has to follow attributes (ii) and (iii).

Definition 2. A mobility trajectory $(T_m)_{m=0}^{\infty}$ reaches a steady state in l time periods ($l \in \mathbb{N}$) if $T_{l-1} \neq T_l$ and $T_i = T_l$ for $i \geq l$. (Although it is quite obvious that when $i = l$, $T_i = T_l$, resorting to a weak inequality is more accommodating, as can be seen in the next example.)

Example 1

Let $n = 4$ and $k = 3$. Then there exist eight possible mobility trajectories. In each of these cases, the initial distribution is one in which the four "near-sighted" individuals reside in region A, namely condition (i) above, $T_0 = (\{1, 2, 3, 4\}, \varnothing, \varnothing)$, holds. Then:

(i) $T_1 = (\{4\}, \{3, 2, 1\}, \varnothing)$ and $T_i = (\{4\}, \{3\}, \{2, 1\})$ for $i \geq 2$.

(ii) $T_1 = (\{4\}, \{3, 2\}, \{1\})$ and $T_i = (\{4\}, \{3\}, \{2, 1\})$ for $i \geq 2$.

(iii) $T_i = (\{4\}, \{3, 1\}, \{2\})$ for $i \geq 1$.

(iv) $T_i = (\{4\}, \{2, 1\}, \{3\})$ for $i \geq 1$.

(v) $T_1 = (\{4\}, \varnothing, \{3, 2, 1\})$ and $T_i = (\{4\}, \{2, 1\}, \{3\})$ for $i \geq 2$.

(vi) $T_1 = (\{4\}, \{1\}, \{3, 2\})$ and $T_i = (\{4\}, \{2, 1\}, \{3\})$ for $i \geq 2$.

(vii) $T_i = (\{4\}, \{2\}, \{3, 1\})$ for $i \geq 1$.

(viii) $T_i = (\{4\}, \{3\}, \{2, 1\})$ for $i \geq 1$.

Three characteristics stand out. First, each of the eight trajectories of the mobility dynamics reaches a steady state. Second, the number of time periods that it takes to reach a steady state is one or two. Third, there are multiple steady states. Naturally, the following questions arise: do the mobility trajectories for *any* $n \geq 4$ and $k \geq 3$ reach a steady state? And if they do, what will be the maximal number (the upper bound) of the time periods that it will take to reach a steady-state distribution of the individuals across the regions? In the case of $n = 4$ and $k = 3$ of Example 1 we have seen that the maximal number is two. As will be shown in our first claim, it is also possible, albeit more challenging, to identify the maximal number of time periods that it will take to reach a steady-state distribution in the case of any $n \geq 4$ and $k \geq 3$.

Claim 1. Let there be $n \geq 4$ "near-sighted" individuals, and let there be $k \geq 3$ regions. Then a steady state is reached. In the steady state, the difference between the numbers of individuals in the regions is at most one. The biggest (maximal) number of time periods that it takes to reach a steady state is $\left\lfloor \dfrac{k-1}{k} n \right\rfloor$.

Proof. The proof is in the appendix.

In the remainder of this sub-section we show how the formula $\left\lfloor \dfrac{k-1}{k} n \right\rfloor$ can be obtained heuristically. A consideration to bear in mind is that in order to "hit" the largest number of time periods that it will take to reach a steady state, when individuals face multiple, equally-attractive mobility possibilities, the individuals elect to move to the same region all in the same time period.

In time period 1, individual n, who has nothing to gain from moving, stays in region A_1, whereas all the other individuals move to another region, say to region A_2 so as to obtain a higher rank. Next, each of the individuals $n-2, n-3, n-4, \ldots, 1$ reasons that he can obtain a higher rank if he moves to yet another region, that is, to a region other than A_1 and A_2. Thus, in time period 2, individuals $n-2, n-3, n-4, \ldots, 1$ move to the same region, say to region A_3, so that in time period 2 the distribution of the individuals is n in region A_1, $n-1$ in region A_2, and the remainder of the individuals are in region A_3.

(i) If $\dfrac{n}{k} \leq 1$, then in time period $n-1$, all the individuals are evenly distributed between the regions such that every individual has rank 1. A steady-state distribution is reached in $n-1$ time periods because in each time period one individual ceases moving, and because individual n never moves.

(ii) If $1 < \frac{n}{k} \leq 2$, then in time period k all the regions will be inhabited by one individual each, and individual $n-k$ will elect to stay in region A_k. The remainder of the individuals will continue moving from one region to another in "a herd" fashion as described above, until the difference between the numbers of individuals in the regions is at most one. However, because exactly one individual ceases moving in every time period other than time period k, because in time period k two individuals cease moving, and because individual n never moves, a steady-state distribution is achieved in $n-2$ time periods.

(iii) If $2 < \frac{n}{k} \leq 3$, then the same procedure as before is in place: one individual ceases moving in every time period other than k and $2k-1$, two individuals cease moving in time periods k and $2k-1$, and individual n never moves. As a result, a steady-state distribution is achieved in $n-3$ time periods.

We thus see a pattern: when individuals move in the manner described above and $\frac{n}{k}$ is an integer, then the general formula of the maximal number of time periods that it takes to reach a steady state is $n - \frac{n}{k}$. Clearly, n need not be a multiple of k. But then (namely when $\frac{n}{k}$ is not an integer), we can easily adjust the $n - \frac{n}{k}$ formula by applying the floor function to $n - \frac{n}{k}$ so that the maximal number of time periods that it takes to reach a steady state is $\left\lfloor n - \frac{n}{k} \right\rfloor$. An example is in (the caption of) Figure 1. By a simple transformation we get that $\left\lfloor n - \frac{n}{k} \right\rfloor = \left\lfloor \frac{kn-n}{k} \right\rfloor = \left\lfloor \frac{k-1}{k} n \right\rfloor$, where the last term is as stated in the claim.

In Figure 1, the preceding account is illustrated by means of an example of a "maximal" trajectory.

A	B	C		A	B	C		A	B	C		A	B	C		A	B	C		A	B	C
8				8				8				8				8				8		
7					7				7				7				7				7	
6					6					6				6				6				6
5			$\rightarrow$		5		$\rightarrow$			5	$\rightarrow$			5	$\rightarrow$			5	$\rightarrow$			5
4					4					4		4				4				4		
3					3					3		3					3				3	
2					2					2		2					2				2	
1					1					1		1					1					1

Figure 1. "Near-sighted" individuals: An example of a "maximal near-sighted" mobility trajectory $\left(\widehat{T}_m\right)_{m=0}^{5}$ for $n=8$ and $k=3$. A steady-state distribution is reached in $\left\lfloor \frac{k-1}{k} n \right\rfloor = \left\lfloor \frac{2}{3} n \right\rfloor = 5$ time periods, and $\widehat{T}_k = \widehat{T}_5$ for $k > 5$.

2.4 Four or more individuals: The case of "far-sighted" individuals

In the "near-sighted" setting, individuals do not take into account the actions of others; they assume that in a subsequent time period all the other individuals will stay in the region in which they happen to be located. In this sub-section we consider instead individuals who are "far-sighted:" as already noted in Sub-section 2.1, in contemplating their mobility behavior, these individuals figure out and take into account the decisions that will be taken by other individuals who are higher up in the hierarchy of the income distribution. In other words, the assumption that the individuals base their mobility decisions on the observed current state without forming expectations as to how other individuals will behave simultaneously can also be relaxed.

When individuals do form such expectations, we characterize them as "far-sighted."[4] In terms of the number of time periods it takes to reach a steady-state distribution and in terms of the steady-state distribution itself, it is easy to see the differences between "near-sightedness" and "far-sightedness" by looking at the simple case of four individuals and three regions.

Example 2

We use here the same notation that we used in Sub-section 2.3. Let $n = 4$ and $k = 3$. In time period 0, the four "far-sighted" individuals reside in region A. Because individual 4 knows that in any region he will be ranked top, he has no reason to move from region A. Individual 3, being "far-sighted," is aware of individual 4's reasoning and, hence, knows that by moving to either region B or region C he will definitely raise his rank. Therefore, in time period 1 he moves to one of these regions and stays there. Individual 2, being "far-sighted," is aware of the reasoning of individuals 4 and 3, and thus he knows that if he moves away from region A and manages to avoid being in the same region as individual 3, he will be ranked top in his region. Otherwise, he will be ranked second in his region. Therefore, in time period 1 he moves either to region B or to region C, and in time period 2 he moves to the other of these two regions if he finds out that he happens to be in the same region as individual 3; otherwise, he stays in his region. After time period 2, he is ranked top in his region, so he does not move again. Individual 1 can reason similarly to individuals 4, 3, and 2 and thus knows that in the "long term" (namely after two time periods or later on), there will be exactly one higher ranked individual in each of regions A, B, and C, so he will be ranked second in any region. Therefore, he cannot improve his rank by moving from region A, so he stays there.

Summing up: from the initial distribution $\theta_0 = (\{1, 2, 3, 4\}, \varnothing, \varnothing)$, four "far-sighted" mobility trajectories $(\theta_m)_{m=0}^{\infty} = (X_1^m, X_2^m, \ldots, X_k^m)_{m=0}^{\infty}$ can emerge:

(i) $\theta_1 = (\{4, 1\}, \{3, 2\}, \varnothing)$ and $\theta_i = (\{4, 1\}, \{3\}, \{2\})$ for $i \geq 2$.

(ii) $\theta_1 = (\{4, 1\}, \varnothing, \{3, 2\})$ and $\theta_i = (\{4, 1\}, \{2\}, \{3\})$ for $i \geq 2$.

[4] A formal definition of the "far-sighed" mobility dynamics turns out to be long and tedious, while not conferring additional insight. We have therefore elected not to include this definition in the paper. However, we have constructed such a definition, and it is available on request.

(iii) $\theta_i = (\{4, 1\}, \{3\}, \{2\})$ for $i \geq 1$.

(iv) $\theta_i = (\{4, 1\}, \{2\}, \{3\})$ for $i \geq 1$.

The basic characteristics are the same as in the case of the "near-sighted" dynamics of Example 1: each of the trajectories of the mobility dynamics reaches a steady state in one or two time periods. However, we can also see the features that distinguish between the "far-sighted" setting and the "near-sighted" setting. First, in the "far-sighted" setting there are fewer possible trajectories: four rather than eight. Second, in the "far-sighted" setting individuals are more likely to stay where they are: for example, whereas in time period 1 of the "near-sighted" dynamics every individual other than individual 4 moves from region A, in the "far-sighted" dynamics individual 1 stays in region A. As a consequence, the steady states differ: individual 1 is in region A in every steady state if he is "far-sighted" whereas he does not end up in region A if he is "near-sighted."

A "near-sighted" individual *experiences*, so to speak, the whereabouts of higher ranked individuals and then adjusts to that. A "far-sighted" individual *anticipates* the movement of higher ranked individuals and then adjusts to that. As a result of this difference, convergence to a steady-state distribution of "far-sighted" individuals between the regions requires fewer time periods than convergence to a steady-state distribution of the "near-sighted" individuals. Intuitively, if we could prove convergence to a steady-state distribution in the case of "near-sightedness" - a distribution that individuals with a limited capacity to figure out what is best for them still manage to reach - then it is quite reasonable to expect that convergence to a steady-state distribution will be reached when individuals have a greater capacity for this. These considerations simplify considerably the treatment of a general "far-sightedness" case, to which we attend next.

To begin with, and as before, we assume that all the individuals know that individual n will not move, so other than him, all of them move. We show that in this "far-sightedness" setting, a steady state is reached in no more than $k - 1$ time periods.

Claim 2. Let there be $n \geq 4$ individuals who are "far-sighted," and let there be $k \geq 3$ regions. Then a steady state is reached as follows. Individuals $n - kl$ for any $l \in \mathbb{N}$ do not move, while all the other individuals move. These other individuals are distributed between regions $A_2, A_3, \ldots, A_k$ in such a way that each of the individuals $n - kl - (k - 1)$, $n - kl - (k - 2)$ up to $n - kl - 1$ ends up in a different region. This steady state is obtained after no more than $k - 1$ time periods. In the steady state, the difference between the numbers of individuals in the regions is at most one.

Proof. The proof is in the appendix.

Example 3

Let $k = 3$. Then in steady state, the number of individuals in regions A, B, and C are, respectively, as follows.

When $n = 3q$, $q \in \mathbb{N}$: q, q, and q.

When $n = 3q + 1$, $q \in \mathbb{N}$: $q + 1$, q, and q.

And when $n = 3q + 2$, $q \in \mathbb{N}$: either $q + 1$, q, and $q + 1$; or $q + 1$, $q + 1$, and q.

3. An economy of three or more regions: Social welfare

As noted in the Introduction, in assessing social welfare we follow the measure introduced in Stark (2017).

Definition 3. Social welfare under rank preferences is the negative of the sum of the ranks of the individuals.

For example, and as illustrated in the Introduction, when two individuals with different incomes are in region A, then the sum of the first rank of one of the individuals and of the second rank of the other individual is $1 + 2 = 3$; when the lower-ranked individual moves to empty region B, then the sum of the ranks is $1 + 1 = 2$. Social welfare in the latter case at (-2) is higher than social welfare in the former case at (-3).

Claim 3. The objective of bringing social welfare under ordinal preferences (rankings) to a maximum is achieved upon any distribution of the individuals in which they are arranged evenly or evenly but for one between the regions. Both steady-state outcomes (for "near-sighted" individuals and for "far-sighted" individuals) yield the social optimum.

Proof. The proof is in the appendix.

We note that the socially optimal distributions are also Pareto-efficient: under a socially optimal distribution, no individual can improve his rank without lowering the rank of at least one other individual. Otherwise it would be possible to improve the level of social welfare of a distribution by improving the rank of an individual, which contradicts the notion of optimality. Therefore, by Claim 3, all steady-state outcomes for both types of dynamics are Pareto-efficient.

However, a Pareto-efficiency of a distribution does not imply that the distribution is socially optimal, nor that it is a steady-state distribution. For example, when there are three regions then the distribution of $n = 8$ individuals $\{\{8, 1\}, \{7, 2\}, \{6, 5, 4, 3\}\}$ is Pareto-efficient (if any individual moves and thereby improves his rank, the rank of at least one other individual deteriorates), but this distribution is neither socially optimal nor a steady state for any of our representations of dynamics. (Recall Figure 1.) Therefore, the results that we obtain are "stronger" than a mere Pareto-efficiency of steady states.

Example 4

Let $k = 3$. Then in a steady state, the level of social welfare is as follows.

When $n = 3q$, $q \in \mathbb{N}$:
$s(X_A, X_B, X_C) = -\frac{3}{2}(q^2 + q)$.

When $n = 3q + 1$, $q \in \mathbb{N}$:
$s(X_A, X_B, X_C) = -\frac{1}{2}(q^2 + 3q + 2) - (q^2 + q) = -\frac{3}{2}q^2 - \frac{5}{2}q - 1$.

And when $n = 3q + 2$, $q \in \mathbb{N}$:
$s(X_A, X_B, X_C) = -(q^2 + 3q + 2) - \frac{1}{2}(q^2 + q) = -\frac{3}{2}q^2 - \frac{7}{2}q - 2$.

4. Complementary reflections

To discover that the sum total of the behavior of individuals whose actions are guided by a desire to improve their own wellbeing yields the optimal social outcome is an intriguing result. When individuals act in such a manner so as to improve their wellbeing without taking into consideration the consequences of their actions for the wellbeing of others, social welfare is not expected to be maximal. After all, and typically, selfish behavior is the source of negative externalities, such that the gains of some are the pain of others. We showed that regardless of whether the choice of region is by "near-sighted" individuals or by "far-sighted" individuals, the choice results in a steady-state distribution of the population, and that this distribution maximizes social welfare. These findings align with results reported in Stark's (2017) model of two regions. The analysis conducted in the current paper informs us that the earlier results are not the outcome of a relatively simple choice of location between two regions. In short, the main results obtained in the current paper are that as in the original analysis of two regions, a steady-state distribution is reached, and that the steady-state distribution maximizes social welfare.

Reaching a steady-state distribution and obtaining optimal social welfare can occur even when mobility is not cost-free. We show this by means of a simple example. Suppose that the cost, c, of every move is $0 < c < 1$, and that the benefit from gaining one rank is 1, that the benefit from gaining two ranks is 2, and so on. Suppose that there are four "far-sighted" individuals whose incomes, and hence ranks, are 4, 3, 2, and 1, and that there are three regions: A, B, and C. To begin with, if moving is costless, then in the first time period individual 4 stays in region A, and the other three individuals move to empty regions B and C. Suppose that they move in such a way that individuals 3 and 1 are in region B, and individual 2 is in region C. No one then has an incentive to move any further; a steady-state distribution is reached in which two regions are inhabited by one individual each, and one region is inhabited by two individuals. The three moving individuals gained: individual 3 gained one rank, individual 2 gained two ranks, and individual 1 gained two ranks. If, however, the cost of move is $0 < c < 1$, then the same moves will occur; the gain to each of the three individuals is bigger than the cost. A positive yet relatively small cost of a move can leave the prediction of the cost-free setting as is.

Reaching a steady-state distribution need not occur when absolute income matters too, yet a rank consideration matters more. Let there be two individuals: n and $n - 1$ (n is always wealthier than $n - 1$), let there be two regions, A and B, and let it be the case that because of a synergy effect, both individuals will be more productive (and, thus, wealthier) when they are in the same region (working together) than when they are in separate regions (operating alone). In such a setting, individual n will prefer to be in the same region as individual $n-1$, say in region A. However, if for individual $n-1$ the displeasure from having a worse wealth rank (when he is in the same region as individual n rather than when he is in a region all by himself) outweighs the pleasure gained from having a higher level of wealth, then individual $n-1$ will prefer to be in region B. Individual n will then prefer to move to region B himself. But then, individual $n-1$ will be better off by moving to region A. And so on. While individual $n-1$ "runs away" from individual n, individual n "runs after" individual $n-1$; a never-ending chase results, with no convergence to a steady state.

The essential results reported in the preceding sections do not depend on the attribute that, to begin with, all the individuals reside in the same region. Even when we discard part (i) of Definition 1, then every "near-sighted" trajectory reaches a steady state such that the difference between the numbers of the individuals in the regions is at most one.[5] Also, Claims 2 and 3 and their proofs are the same for different starting distributions.[6] Quite intuitively, the only difference is that the maximal number of time periods needed to reach a steady state might be smaller when to begin with, not all the individuals reside in the same region. We know that at a steady state the individuals are distributed between the regions as equally as possible. Therefore, there is no initial distribution that is farther from a steady state than the one we considered in the preceding sections; under more "favorable" initial conditions, convergence to a steady state can only be faster. We end this section with $k=3$ and $n=8$ configurations that help seeing this result.

Example 5

From Claim 1 and Figure 1, we know that the trajectory of "herd movement to the next available region" in the case of "near-sighted" individuals for $k=3$ and $n=8$ reaches a steady state after five time periods. In general, when we discard the attribute that to begin with, all the individuals reside in the same region, we may just as well end up lowering the number of time periods needed to reach a steady state, possibly all the way down to 0. For example, if the starting distribution of the eight individuals in the three regions is $(X_A^0, X_B^0, X_C^0) = (\{7, 4, 2\}, \{8, 3, 1\}, \{6, 5\})$, then no individual has an incentive to move, so we are in a steady state right from the start.

Another possibility is illustrated in Figure 2 where we assume that to begin with, even-numbered individuals are in region A and that odd-numbered individuals are in region B. Then, in time period 1, all the individuals other than 8 and 7 move to region C, so the resulting distribution is the same as the one that we obtained after two time periods of a "maximal" trajectory when the starting distribution was of all the individuals residing in region A (recalling Figure 1). When the initial distribution of the individuals is as per Figure 2, the time that it takes to reach a steady state is

[5]The reasoning that leads to this conclusion is the same as the one presented in the proof of Claim 1 for the case in which, to begin with, all the individuals are in region A_1.

[6]With regard to Claim 2 we need merely to add the assumption (which we can make without any loss of generality) that to begin with, individual n resides in region A_1.

shorter by one time period compared to when the initial distribution is as portrayed in Figure 1.

A	B	C		A	B	C		A	B	C		A	B	C		A	B	C
8				8				8				8				8		
	7				7				7				7				7	
6						6				6				6				6
	5		→			5	→			5	→			5	→			5
4						4		4				4				4		
	3					3		3					3				3	
2						2		2					2				2	
	1					1		1					1					1

Figure 2. "Near-sighted" individuals: An example of a "maximal near-sighted" mobility trajectory for $n = 8$ and $k = 3$ when the initial distribution of the individuals is $(X_A^0, X_B^0, X_C^0) = (\{8,6,4,2\}, \{7,5,3,1\}, \varnothing)$. A steady-state distribution is reached in four time periods.

That being said, we hasten to add that for some initial distributions, the maximal number of time periods needed to reach a steady state can remain the same as when the initial distribution is such that all the individuals reside in the same region. Considering once again $k = 3$ and $n = 8$, we illustrate this possibility in Figure 3.

A	B	C		A	B	C		A	B	C		A	B	C		A	B	C		A	B	C
8				8				8				8				8				8		
7					7				7				7				7				7	
6					6					6				6				6				6
5			→		5		→			5	→			5	→			5	→			5
4					4					4		4				4				4		
3					3					3		3					3				3	
2					2					2		2					2				2	
	1				1					1		1					1					1

Figure 3. "Near-sighted" individuals: An example of a "maximal near-sighted" mobility trajectory for $n = 8$ and $k = 3$ when the initial distribution of the individuals is $(X_A^0, X_B^0, X_C^0) = (\{8,7,6,5,4,3,2\}, \{1\}, \varnothing)$. A steady-state distribution is reached in five time periods.

In Figure 3, we assume that to begin with, all the individuals but for individual 1 reside in region A, and that individual 1 resides in region B. If we follow the rules of "herd mobility to the next region," then in time period 1, all the individuals from region A other than individual 8 move to region B so that the resulting distribution is the same as the one that we obtained after one time period of the "maximal" trajectory when the starting distribution was of all the individuals residing in region A (recalling Figure 1). When the initial distribution of the individuals is as per Figure 3, the time that it takes to reach a steady state is the same as when the initial distribution is as portrayed in Figure 1.

The assumption that the regions are identical is the last assumption that we might consider revoking. Suppose that in selecting the region to which to move, randomness is replaced by a "convention:" the individual who assesses the prospect of having the same rank in different regions chooses to move to the region with the lowest label, namely he prefers residing in A_1 over A_2, in A_2 over A_3, and so on, as long as the rank that he will obtain is the same. In the example of four "far-sighted" individuals and three regions, A, B, and C, while individuals 4 and 1 do not move, individual 3 will be known to move to region B. Aware of the convention that leads individual 3 to move to region B, individual 2 will move to region C. In such a case, a steady-state distribution will be reached in one time period with, to recapitulate, individual 3 residing in region B, and individual 2 residing in region C. In general, for any number of individuals and regions, "far-sighted" individuals will reach a steady-state distribution in one time period because from the very beginning they will be able to predict where everyone else will end up residing, so they can choose their optimal region right at the outset.

In the case of four "near-sighted" individuals and three regions, in time period 1 individuals 3, 2, and 1 will move to region B (according to the "convention" they will prefer it to region C), leaving individual 4 in region 1. Then, in time period 2, individuals 2 and 1 will move to region C (while individuals 4 and 3 do not move), and in time period 3 individual 1 will move to region A. Then, the "near-sighted" dynamics reaches a steady state {{4, 1}, {3}, {2}} after three time periods. In general, for any number of individuals n and any number of regions k, in time period $l < n-1$ individuals $n, n-1, \ldots, n-l+1$ remain in the regions in which they currently reside, while individuals $n-l, n-l-1, \ldots, 1$ move to the next region in a circular order $\ldots \to A_1 \to A_2 \to \ldots \to A_k \to A_1 \to \ldots$. Therefore, the steady state will be reached after $n-1$ time periods, when individual 1 ceases to move.

A "convention" scenario is not, however, in the spirit of our perception that regions are identical in all relevant respects, and are equally close by or are equally distant so that none can be prioritized.

5. Concluding remarks

We can think of three interesting ways to obtain additional insights from our analysis.

First, we have an input into migration theory. Replacing move with migrate and mobility trajectory with migration trajectory takes us to a theory of rank-seeking migration when the region of choice is one of many. As already shown a long time ago (Stark, 1993), the motives for migration are not only cardinal; they are ordinal too: comparisons with others influence location choices. Rank-seeking belongs to the ordinal category. The approach undertaken in this paper is to develop a pure theory (improvement of rank is all that matters), as doing this serves to flesh out key features. In less abstract settings, a utility representation will incorporate terms of both types, say income and rank, allowing for tradeoffs and different weighing across individuals and cultures.[7]

[7] Starting with Stark and Taylor (1991) all the way through to Kafle et al. (2020), a large number of empirical studies have shown that relative deprivation, defined as the aggregate of the income excesses that

Second, suppose that we were to define an index of ordinal inequality, an "ordinal Gini coefficient" of sorts, and endow that index with maximal and minimal values. For example, if there are three individuals and three regions, and when, to begin with, the three individuals are in the same region, then the negative of the sum of the levels of their rank deprivation is (-3). This sum can be normalized as value 1 of the index of ordinal inequality. When the individuals reach the steady-state distribution in which they reside one individual per region, then rank deprivation, and thereby the value of the index of ordinal inequality, are nil. These two numbers are, respectively, the maximal and the minimal values of that ordinal inequality index. When there are more individuals than regions, we have shown that the steady state obtained at which the individuals are arranged evenly or evenly but for one between the regions confers the highest level of social welfare. This level of social welfare is the flip side of the minimal level of the inequality index - it yields the lowest achievable level of ordinal inequality. The route from initial maximal inequality to steady-state minimal inequality is not monotonic (there can be intervening time periods in which ordinal inequality increases), but the final state always yields the lowest inequality. For students of inequality, an appealing feature of the dynamics of our analysis is that the sum total of the behavior of the individuals yields not only the maximal level of social welfare but also the minimal level of rank inequality.

A third interesting way of obtaining additional insight from our analysis would be to use it "upside down;" in a way, we could learn about the capacity of people's brains by watching where their legs take them: in order to differentiate between "near-sightedness" and "far-sightedness," we do not need to detect neural responses using functional magnetic resonance imaging (fMRI). Instead, we can look at the steady-state distribution of the individuals between regions. Because the patterns of the equilibria distributions differ, we can draw on the one-to-one relationship between a pattern and the type of "sightedness" to form judgment about the individuals' mental capacities. To illustrate, recalling the simple case of four individuals and three regions, if we observe a steady-state distribution in which individual 1 resides in region B or in region C (for example, $\{\{4\}, \{3\}, \{2, 1\}\}$), then we will infer "near-sightedness;" and if we observe a steady-state distribution such that individual 1 resides in region A (for example, $\{\{4, 1\}, \{3\}, \{2\}\}$), then we will infer "far-sightedness."

---

individuals experience within their reference group divided by the size of the group, exerts an independent and significant impact on their migration from the reference group. This empirical record can serve as a constructive hint that a novel theory of migration as rank-seeking behavior could successfully be taken to the data. A bridge between the theory and an empirical implementation is not difficult to construct. For example: an individual likes absolute income and dislikes rank deprivation, and assigns to these two terms in his utility function the weights of α and $-(1-\alpha)$, respectively, where $\alpha \in (0, 1)$. Thus, the individual's utility function can be represented by $u(x, RD) = \alpha x - (1-\alpha)RD$, where x denotes the individual's income, and RD denotes his rank deprivation. One testable hypothesis will be that the higher the (absolute value of the) weight accorded to RD, the closer will be the observed steady-state distribution to the steady-state distribution predicted by the theory.

Appendix: Proofs of Claims 1, 2, and 3

Proof of Claim 1. For ease of reference, we replicate here the claim.

Claim 1. Let there be $n \geq 4$ "near-sighted" individuals, and let there be $k \geq 3$ regions. Then a steady state is reached. In the steady state, the difference between the numbers of individuals in the regions is at most one. The biggest (maximal) number of time periods that it takes to reach a steady state is $\left\lfloor \frac{k-1}{k} n \right\rfloor$.

Prior to presenting the proof itself, we introduce two definitions, present an example, and state and prove a supportive lemma.

Definition A1. Let $n \geq 4$, $k \geq 3$ and $(T_m)_{m=0}^{\infty}$ be a "near-sighted" mobility trajectory. We say that individual i is settled in time period m if for each individual $j \geq i$ there exists $\alpha(j) \in \{1, 2, \ldots, k\}$ such that $j \in X^t_{\alpha(j)}$ for $t \geq m$. We then denote the smallest number m such that individual i is settled in time period m by $st(i)$, and we say that individual i becomes settled in time period $st(i)$.

It may seem that individual i is settled when he no longer moves, namely in time period m for which there exists $\alpha \in \{1, 2, \ldots, k\}$ such that $i \in X^t_\alpha$ for $t \geq m$, regardless of the behavior of any other individual. However, as long as all the individuals who are initially ranked higher than individual i keep moving, individual i cannot be sure about his rank and, thus, whether or not he finds it desirable to move, meaning that he cannot as yet be settled. We illustrate this observation with the help of Figure A1.

A	B	C
6		
5		
4		
3		
2		
1		

$\rightarrow$

A	B	C
6		
	5	
		4
		3
		2
	1	

$\rightarrow$

A	B	C
6		
	5	
		4
		3
2		
	1	

Figure A1. "Near-sighted" individuals: A "near-sighted" trajectory for $n = 6$ and $k = 3$ (the path taken until the individuals reach their steady-state regions).

By Definition A1, in the trajectory depicted in Figure A1, individual 1 is not yet settled in time period 1, even though subsequently he does not move away from the region to which he moved in time period 1. The reason is that while he will not be moving himself, he needs to pay attention to the mobility behavior of individual 2 who has not as yet ceased moving. It is still possible that in time period 2 individual 2 will move to region B, which will prompt individual 1 to move again. Only in time period 2 when individual 2 is settled in region A can individual 1 be confident that he will not need to move further and, thus, we will be able to say that he is settled. In this case then, $st(1) = 2$.

Definition A2. For $m \geq 0$, we denote by $ST(m)$ the *set* of individuals who are settled in time period m, and we denote by ST_m the *number* of individuals who are settled in time period m, namely:

$$ST(m) \equiv \{i \in \{1, 2, \ldots, n\} : st(i) \leq m\}$$

and[8]

$$ST_m \equiv |ST(m)|.$$

Naturally, when $j > i$ and $i \in ST(m)$, then $j \in ST(m)$. Consequently,

$$ST(m) \equiv \{n - ST_m + 1, n - ST_m + 2, \ldots, n\}.$$

Thus, a trajectory reaches a steady state in m time periods when m is the smallest number such that $ST_m = n$.

Example A1

In the trajectory depicted in Figure A1: $st(6) = 0$, $st(5) = st(4) = st(3) = 1$, and $st(2) = st(1) = 2$. Thus: $ST(0) = \{6\}$ and $ST_0 = 1$; $ST(1) = \{3, 4, 5, 6\}$ and $ST_1 = 4$; and $ST(m) = \{1, 2, 3, 4, 5, 6\}$ and $ST_m = 6$ for $m \geq 2$. We see that this trajectory reaches a steady state in two time periods because $m = 2$ is the smallest number for which $ST_m = 6$.

The following lemma lists properties of the sequences $(ST(m))_{m=0}^{\infty}$ and $(ST_m)_{m=0}^{\infty}$.

Lemma A1. Let $n \geq 4$, $k \geq 3$ and $(T_m)_{m=0}^{\infty}$ be a "near-sighted" mobility trajectory, $a_0 = ST_0$ and $a_m = ST_m - ST_{m-1}$ for $m \geq 1$ (namely a_m is the number of individuals who become settled in time period m). Then:

(i) $ST(0) = \{n\}$.

(ii) If $ST_m < n$, then $ST_{m+1} > ST_m$.

(iii) $|ST(m) \cap X_{\alpha}^{m}| - |ST(m) \cap X_{\beta}^{m}| \leq 1$ for every $\alpha, \beta \in \{1, 2, \ldots, k\}$ and every $m \geq 0$.

(iv) If $ST_m \neq n$, then ST_m is not a multiple of k for any $m \geq 0$.

(v) If $ST_{m-1} < n$, then

$$m + 1 + \left\lfloor \frac{m}{k-1} \right\rfloor \leq ST_m. \quad (1)$$

[8] $|X|$ denotes the number of elements in a set X.

Proof. We prove separately each of the five parts of the lemma.

(i) Naturally, individual n never moves because he already occupies rank 1 in region A_1, so $n \in ST(0)$. All the other individuals move in the first time period because rank 1 in regions $A_2, A_3, \ldots, A_k$ is vacant while their rank in A_1 is lower than 1. Thus, if $j \neq n$, then $j \notin ST(0)$. Therefore, $ST(0) = \{n\}$.
(ii) By definition, $ST(m) \subset ST(m+1)$ (and $ST_{m+1} \geq ST_m$), so we only need to show that $ST(m) \neq ST(m+1)$ (and, thus, $ST_{m+1} \neq ST_m$).

We assume that $ST_m = l < n$. Then individual $n - l \notin ST(m)$. However, all the individuals numbered higher than $n - l$ are settled in time period m, so that in time period $m + 1$ individual $n - l$ chooses the region where he is ranked highest and moves there. After that, he cannot improve his rank by moving further, so he is settled in time period $m + 1$, namely $n - l \in ST(m + 1)$. Hence, $ST(m) \neq ST(m + 1)$, and $ST_m < ST_{m+1}$.
(iii) We assume, by contradiction, that there exist $m \geq 0$ and regions $\alpha, \beta \in \{1, 2, \ldots, k\}$ such that

$$|ST(m) \cap X^m_\alpha| - |ST(m) \cap X^m_\beta| > 1.$$

Let $x = \min(ST(m) \cap X^m_\alpha)$ and $|ST(m) \cap X^m_\beta| = l$. In particular, $R(x, X^m_\alpha) \geq l + 2$. Moreover, $x \in ST(m)$, thus $x > j$ for every $j \in X^m_\beta \setminus ST(m)$, so that $R(x, X^m_\beta \cup \{x\}) \leq l + 1$. Therefore, x will gain rank by moving from region α to region β in time period $m + 1$, which contradicts the presumption that $x \in ST(m)$. Thus, assuming that $|ST(m) \cap X^m_\alpha| - |ST(m) \cap X^m_\beta| > 1$ is false.
(iv) We assume, again by contradiction, that $ST_m \neq n$ and $ST_m = kl$ for some $l \in \mathbb{N}$. Then $ST(m) = \{n - kl + 1, \ldots, n\}$. By the preceding part (iii) of this lemma $|ST(m) \cap X^m_1| = |ST(m) \cap X^m_2| = \ldots = |ST(m) \cap X^m_k| = l$. Thus, if $p = n - 3l$, then $R(p, X^M_1 \cup \{p\}) = R(p, X^M_2 \cup \{p\}) = \ldots = R(p, X^M_k \cup \{p\}) = l + 1$ for every $M \geq m$. Therefore, no matter in which of the regions $A_1, A_2, \ldots, A_k$ individual p is located in time period m, he will have no incentive to move away from that region. Moreover, all the individuals who are numbered higher than p belong to $ST(m)$, thus also $p \in ST(m)$, which contradicts the assumption $ST_m = kl$. Therefore, ST_m is not a multiple of k.
(v) By definition, $ST_m = \sum_{i=0}^{m} a_i$ for $m \geq 0$. We assume that $m \geq 0$ is such that $ST_{m-1} < n$. By part (i) of the lemma, $a_0 = 1$. By part (ii) of the lemma, $(ST_i)_{i=0}^{m}$ is an increasing sequence of integers, thus $a_i \geq 1$ for $m \geq 1$. We assume that there exists $i \geq 0$ such that $a_{i+1} = a_{i+2} = \ldots = a_{i+k-1} = 1$. Then, $ST_i, ST_{i+1}, \ldots, ST_{i+k-1}$ are k consecutive integers, so one of them is a multiple of k, which contradicts point (iv) of the lemma. Therefore, for $i \geq 0$, among $k - 1$ consecutive elements of the sequence $(a_{i+1}, a_{i+2}, \ldots, a_{i+k-1})$ there exists at least one which is not smaller than 2. Thus, at least $\left\lfloor \dfrac{m}{k-1} \right\rfloor$ elements of the sequence $(a_i)_{i=0}^{m}$ are not smaller than 2, and the remainder elements are not smaller than 1 (by point (ii) of the lemma). Therefore:

$$ST_m = \sum_{i=0}^{m} a_i \geq \left(m + 1 - \left\lfloor \frac{m}{k-1} \right\rfloor\right) \cdot 1 + \left\lfloor \frac{m}{k-1} \right\rfloor \cdot 2 = m + 1 + \left\lfloor \frac{m}{k-1} \right\rfloor.$$

Q.E.D.

Proof of Claim 1. The fact that a steady state is reached is an immediate consequence of parts (i) and (ii) of Lemma A1, of $ST_0 = 1$, and of the attribute that the sequence $(ST_m)_{m=0}^{\infty}$ is strictly increasing as long as its elements are smaller than n. Also, the elements of $(ST_m)_{m=0}^{\infty}$ are integers, thus $ST_m \geq m+1$ as long as $ST_{m-1} < n$. Therefore, $ST_{n-1} = n$, namely all the individuals are settled after at most $n-1$ time periods, so a steady state is reached.

The fact that in a steady state the difference between the numbers of individuals in the regions is at most one follows immediately from part (iii) of Lemma A1: if a steady state $(X_1, X_2, \ldots, X_k)$ is reached in time period m, then $(X_1, X_2, \ldots, X_k) = (X_1^m, X_2^m, \ldots, X_k^m)$, and $ST(m) = \{1, 2, \ldots, n\}$. In particular, $X_i^m \cap ST(m) = X_i^m$ for $i = 1, 2, \ldots, k$. Thus, and by part (iii) of Lemma A1, $|X_\alpha| - |X_\beta| = |X_\alpha^m| - |X_\beta^m| = |ST(m) \cap X_\alpha^m| - |ST(m) \cap X_\beta^m| \leq 1$ for every $\alpha, \beta \in \{1, 2, \ldots, k\}$.

We next attend to the third component of the claim, namely to the maximal number of time periods that it takes to reach a steady state. We need to show that there is no "near-sighted" mobility trajectory that requires more than $\left\lfloor \frac{k-1}{k} n \right\rfloor$ time periods to reach a steady state.

To this aim, we assume the contrary. We know that each trajectory reaches a steady state, so there exists $m > \left\lfloor \frac{k-1}{k} n \right\rfloor$ (and $m > \frac{k-1}{k} n$, because m is an integer) such that $ST_{m-1} < n$ and $ST_m = n$. However, by part (v) of Lemma A1,[9]

$$ST_m \geq m + 1 + \left\lfloor \frac{m}{k-1} \right\rfloor = \left\lfloor \frac{km}{k-1} \right\rfloor + 1 \geq \lfloor n \rfloor + 1 = n + 1.$$

Therefore, by contradiction, there exists no "near-sighted" mobility trajectory that requires more than $\left\lfloor \frac{k-1}{k} n \right\rfloor$ time periods to reach a steady state.

It remains to be shown that the upper bound of the number of time periods that are needed to reach a steady state cannot be bigger than $\left\lfloor \frac{k-1}{k} n \right\rfloor$. To this end, following the approach and logic of the heuristic proof in the main text, we construct a "near-sighted" mobility trajectory $\left(\widehat{T}_m\right)_{m=0}^{\infty}$ which reaches a steady state in exactly $\left\lfloor \frac{k-1}{k} n \right\rfloor$ time periods.

Prior to the said construction, we configure a cyclical ordering of the regions: $\ldots \to A_1 \to A_2 \to \ldots \to A_k \to A_1 \to \ldots$. The basic rule of mobility for $\left(\widehat{T}_m\right)_{m=0}^{\infty}$ is as follows. In every time period, in accordance with the notion of "near-sighted"

[9]The first equality in the next line holds because there exist $q, r \in \mathbb{N}$ such that $r \leq k-2$ and $m = (k-1)q + r$, and then $m + \left\lfloor \frac{m}{k-1} \right\rfloor = (k-1)q + r + q = kq + r = \frac{k(k-1)q + (k-1)r}{k-1}$ $= \left\lfloor \frac{k(k-1)q + (k-1)r}{k-1} + \frac{r}{k-1} \right\rfloor = \left\lfloor \frac{km}{k-1} \right\rfloor$.

mobility, as many individuals as possible move from the most populous region, denoted by α, to the adjacent region on the right of region α in this cyclical order, denoted by $\alpha + 1$. We chart the mobility path governed by this rule.[10]

In time period 1, individual n who has nothing to gain from moving, stays in region A_1, whereas all the other individuals who seek to obtain a higher rank move to region A_2.

In time period 2, individuals $n-2, n-3, n-4, \ldots, 1$ move to region A_3, so that in this time period the distribution of the individuals is n in region A_1, $n-1$ in region A_2, and the remainder of the individuals are in region A_3.

In general, in each time period m such that $1 \leq m < \min\{k, n\}$, one individual $(n-m+1)$ stays in region A_m while individuals $n-m, n-3, n-4, \ldots, 1$ move to region A_{m+1}.

If $n \leq k$, then in time period $n-1$, which is when individual 1 moves to region A_n, all the individuals are distributed between the regions such that every individual has rank 1. A steady-state distribution is reached in $n-1$ time periods. The number $n-1$ is derived from the fact that one individual stops moving in every time period, individual n never moves, and a steady state is reached as soon as all the individuals cease moving. Moreover, $n > \frac{k-1}{k}n \geq \left\lfloor \frac{k-1}{k}n \right\rfloor \geq \left\lfloor \frac{n-1}{n}n \right\rfloor = n-1$, thus $n-1$ is the largest integer that is not bigger than $\left\lfloor \frac{k-1}{k}n \right\rfloor$, namely $\left\lfloor \frac{k-1}{k}n \right\rfloor = n-1$, and a steady state is reached in $\left\lfloor \frac{k-1}{k}n \right\rfloor$ time periods. In conclusion, the upper bound of the number of time periods that are needed to reach a steady state cannot be bigger than $\left\lfloor \frac{k-1}{k}n \right\rfloor$ for $n \leq k$.

If $n \geq k+1$, then before time period k, in each of regions $A_1, A_2, \ldots, A_{k-1}$ there is one individual who is not moving any more, and individuals $n-k+1, n-k, \ldots, 1$ are in region A_k. In time period k, two individuals, $n-k+1$ and $n-k$, elect to stay in region A_k and they will not move again, while the remainder of the individuals from region A_k move to region A_1. Then, in each time period m such that $k+1 \leq m < \min\{2k-1, n-2\}$, one individual $(n-m)$ stays in region A_{m-1}, while individuals $n-m-1, n-3, n-4, \ldots, 1$ move to region A_m.

If $k+1 \leq n \leq 2k$, then in time period $n-2$, which is when individual 1 moves for the last time, all the individuals are distributed between the regions such that every individual $i \geq n-k+1$ has rank 1, and each individual $i \leq n-k$ has rank 2. A steady-state distribution is reached in $n-2$ time periods. The number $n-2$ is derived from the fact that one individual ceases moving in every time period other than $k = 1 + (k-1)$, in time period k two individuals cease moving, and individual n never moves. Moreover, $\frac{n-1}{n} > \frac{k-1}{k} > \frac{n-2}{n}$, therefore,

[10] In conjunction with the remainder of this proof, it can be helpful to consult the example in Figure 1, presented at the end of Sub-section 2.3 of the main text.

$n-1 > \frac{k-1}{k}n \geq \left\lfloor \frac{k-1}{k}n \right\rfloor \geq \left\lfloor \frac{n-2}{n}n \right\rfloor = n-2$, thus, $n-2$ is the largest integer that is not bigger than $\left\lfloor \frac{k-1}{k}n \right\rfloor$, namely $\left\lfloor \frac{k-1}{k}n \right\rfloor = n-2$, and a steady state is reached in $\left\lfloor \frac{k-1}{k}n \right\rfloor$ time periods. In conclusion, the upper bound of the number of time periods that are needed to reach a steady state cannot be bigger than $\left\lfloor \frac{k-1}{k}n \right\rfloor$ for $k+1 \leq n \leq 2k$.

If $n \geq 2k$, then before time period $2k-1$, two individuals who are not moving any more reside in each of the regions other than A_{k-1}, and individuals $n-2k+1, n-2k, \ldots, 1$ reside in region A_{k-1}. In time period $2k-1$ two individuals, $n-2k+1$ and $n-2k$, elect to stay in region A_{k-1} and they will not move any further, while the remainder of the individuals from region A_{k-1} move to region A_k.

In general, in each time period m such that $m = 1 + q(k-1)$ for some $q \in \mathbb{N}_+$, two individuals ($n-qk+1$ and $n-qk$) cease moving while individuals numbered lower than $n-qk$ move to the next region in the cyclical ordering. In each time period m such that $2 + q(k-1) \leq m < \min\{1 + (q+1)(k-1), n-(q+1)\}$, one individual ($n-m-q+1$) ceases moving, having been located in some region α, while individuals numbered lower than $n-m-q+1$ move to region $\alpha+1$.

Therefore, if $qk+1 \leq n \leq (q+1)k$ for some $q \in \mathbb{N}_+$, then in time period $n-(q+1)$, which is when individual 1 moves for the last time, all the individuals are distributed between the regions such that the rank of individual i in his region is $\left\lfloor \frac{n-i}{k} \right\rfloor + 1$. A steady-state distribution is reached in $n-(q+1)$ time periods. The number $n-(q+1)$ is derived from the fact that one individual ceases moving in every time period other than $m = 1 + l(k-1)$ for $l = 1, 2, \ldots, q$, in q time periods of the form $m = 1 + l(k-1)$ for $l = 1, 2, \ldots, q$ two individuals stop moving, and individual n never moves. Moreover,

$$\frac{n-q}{n} \geq \frac{q(k-1)+1}{qk+1} > \frac{k-1}{k} = \frac{(q+1)(k-1)}{(q+1)k} \geq \frac{n-(q+1)}{n},$$

therefore, $n-q > \frac{k-1}{k}n \geq \left\lfloor \frac{k-1}{k}n \right\rfloor \geq \left\lfloor \frac{n-(q+1)}{n}n \right\rfloor = n-(q+1)$, thus, $n-(q+1)$ is the largest integer that is not bigger than $\left\lfloor \frac{k-1}{k}n \right\rfloor$, namely $\left\lfloor \frac{k-1}{k}n \right\rfloor = n-(q+1)$, and a steady state is reached in $\left\lfloor \frac{k-1}{k}n \right\rfloor$ time periods. In conclusion, the upper bound of the number of time periods that are needed to reach a steady state in the general case cannot be bigger than $\left\lfloor \frac{k-1}{k}n \right\rfloor$. Q.E.D.

Proof of Claim 2. For ease of reference, we replicate here the claim.

Claim 2. Let there be $n \geq 4$ individuals who are "far-sighted," and let there be $k \geq 3$ regions. Then a steady state is reached as follows. Individuals $n - kl$ for any $l \in \mathbb{N}$ do not move, while all the other individuals move. These other individuals are distributed between regions $A_2, A_3, \ldots, A_k$ in such a way that each of the individuals $n - kl - (k - 1), n - kl - (k - 2)$ up to $n - kl - 1$ ends up in a different region. This steady state is obtained after no more than $k - 1$ time periods. In the steady state, the difference between the numbers of individuals in the regions is at most one.

Proof of Claim 2. To recall, the "fortune" of any individual does not depend on the location of individuals who occupy lower rungs in the incomes hierarchy. The mobility decision taken by individual $n - 1$ depends only on the mobility decision taken by individual n, the mobility decision taken by individual $n - 2$ depends only on the mobility decisions taken by individuals n and $n - 1$, and so on.

In the first period, individual $n - 1$ moves from region A_1 to one of the regions $A_2, A_3, \ldots, A_k$ and never moves again. Individual $n - 2$ moves either to the same region as individual $n - 1$ or to one of the other regions, as he does not know to which region individual $n - 1$ chose to move. In the second time period, individual $n - 2$ moves away if he had chosen the same region as individual $n - 1$ in the first time period, or he stays in the region he resides in if he had chosen any other region in the first time period. We can similarly analyze the behavior of individuals $n - 3, n - 4, \ldots, n - (k - 1)$, and conclude that after no more than $k - 1$ time periods, each of them will reside in one of the regions $A_2, A_3, \ldots, A_k$. Therefore, we know that individual $n - k$ stays in region A_1 because in all other regions there will eventually be one individual ahead of him, so being the second in region A_1 cannot be improved upon by moving to any of the other regions.

The reasoning for individuals $n - kl, n - (kl + 1), \ldots, n - (kl + k - 1)$ for any $l \in \mathbb{N}$ is similar: individual $n - kl$ stays in region A_1 because his rank there is $l + 1$, and because in any of the other regions there will eventually be l individuals ahead of him, so he has nothing to gain from moving. Individual $n - (kl + 1)$ moves and stays in the region that he chose in the first time period. Individual $n - (kl + 2)$ moves in the first time period and either stays or moves in the second time period (depending on whether he chose the same region that $n - (kl + 1)$ chose in the first time period), and so on. The steady-state distribution is reached in just one time period if, for every $l \geq 0$, individuals $n - kl, n - (kl + 1), \ldots, n - (kl + k - 1)$ have chosen different regions, and it is reached in up to $k - 1$ time periods otherwise.

If $n = kq$ for some $q \in \mathbb{N}$, then in the steady state the number of individuals in all k regions is the same (namely, q); the regions are equally populated. If $n = kq + r$ for some $q, r \in \mathbb{N}$ and $1 \leq r \leq k - 1$, then there are $k + 1$ individuals in region A_1 and in $(r - 1)$ regions among $A_2, A_3, \ldots, A_k$, and k individuals in the rest of these regions (namely, $(k - r)$ of them). Q.E.D.

Proof of Claim 3. For ease of reference, we replicate here the claim.

Claim 3. The objective of bringing social welfare under ordinal preferences (rankings) to a maximum is achieved upon any distribution of the individuals in which they are arranged evenly or evenly but for one between the regions. Both steady-state outcomes (for "near-sighted" individuals and for "far-sighted" individuals) yield the social optimum.

Proof of Claim 3. There exist finitely many distributions, thus there exists at least one distribution for which the maximal social welfare is achieved. As before, we let $n \geq 4$ be the number of the individuals, and we let $A_1, A_2, \ldots, A_k$ be the names of the k regions. We recall from Sub-section 2.3 that $(X_1, X_2, \ldots, X_k)$ is a distribution if $X_1, X_2, \ldots, X_k$ are the sets of individuals who reside, respectively, in regions $A_1, A_2, \ldots, A_k$. We denote the level of social welfare that corresponds to such a distribution by $s(X_1, X_2, \ldots, X_k)$. In addition, we denote $|X_j| = a_j$ for $j \in \{1, 2, \ldots, k\}$, where $|S|$ is the number of elements in a set S. The sum of the ranks of the a_j individuals in X_j is $1+2+\ldots+a_j = \dfrac{a_j(a_j+1)}{2}$ for $j \in \{1, 2, \ldots, k\}$. Therefore, the corresponding level of social welfare is

$$s(X_1, X_2, \ldots, X_k) = -\sum_{j=1}^{k} \frac{a_j(a_j+1)}{2} = -\frac{1}{2}\sum_{j=1}^{k}(a_j^2 + a_j).$$

First, we show that any distribution in which the individuals are not arranged evenly or evenly but for one between the regions does not maximize social welfare. We assume that $(X_1, X_2, \ldots, X_k)$ is a distribution in which the individuals are not distributed evenly or evenly but for one between the regions. Without loss of generality, we assume that $a_1 \geq a_2 \geq \ldots \geq a_k$. Then, $a_1 - 1 > a_k$ (because otherwise, the individuals will be distributed evenly or evenly but for one between the regions).

Let $i \in X_1$. We consider the distribution $(X_1 \setminus \{i\}, X_2, \ldots, X_k \cup \{i\})$, namely we modify $(X_1, X_2, \ldots, X_k)$ by moving individual i from region A_1 to region A_k. Then the level of social welfare for distribution $(X_1 \setminus \{i\}, X_2, \ldots, X_k \cup \{i\})$ is:

$$s(X_1 \setminus \{i\}, X_2, \ldots, X_k \cup \{i\}) = -\frac{1}{2}\left[(a_1-1)a_1 + \sum_{j=2}^{k-1}(a_j^2 + a_j) + (a_k+1)(a_k+2)\right]$$

$$= -\frac{1}{2}\left(\sum_{j=2}^{k-1}(a_j^2 + a_j) + a_1^2 + a_k^2 - a_1 + 3a_k + 2\right)$$

$$> -\frac{1}{2}\left(\sum_{j=1}^{k} a_j^2 + \sum_{j=2}^{k-1} a_j - a_1 + 2(a_1 - 1) + a_k + 2\right)$$

$$= -\frac{1}{2}\sum_{j=1}^{k}(a_j^2 + a_j) = s(X_1, X_2, \ldots, X_k).$$

Therefore, $s(X_1, X_2, \ldots, X_k)$ is not a maximal value of social welfare and, consequently, if $(X_1, X_2, \ldots, X_k)$ is a distribution such that the individuals are not distributed evenly or evenly but for one between the regions, then that distribution does not maximize social welfare.

Second, we show that every distribution $(X_1, X_2, \ldots, X_k)$ in which the individuals are arranged evenly or evenly but for one yields the same level of social welfare. To this end, we assume that $(X_1, X_2, \ldots, X_k)$ is a distribution in which the individuals are arranged evenly or evenly but for one between the regions.

There exists a unique pair $(q, r) \in \mathbb{N}_+$ such that $n = kq + r$ and $r \leq k - 1$. Then there are $q + 1$ individuals in r regions and q individuals in $k - r$ regions (otherwise the individuals are not distributed evenly or evenly but for one). Therefore:

$$s(X_1, X_2, \ldots, X_k) = -\frac{1}{2}\sum_{j=1}^{k}(a_j^2 + a_j) = -\frac{r}{2}(q^2 + 3q + 2) - \frac{k - r}{2}(q^2 + q).$$

In particular, when the individuals are distributed evenly or evenly but for one between the regions, the value of the social welfare function depends only on the numbers k, r and q. In turn, the values of r and q depend only on the number of individuals n, and on the number of regions k. In conclusion: for fixed n and k, every distribution in which the individuals are arranged evenly or evenly but for one between regions yields the same level of social welfare. Simultaneously, the yielded social welfare has to be maximal because no distribution in which the individuals are not arranged evenly or evenly but for one between regions can maximize social welfare. From Claims 1 and 2 we know that all the steady-state distributions are among the socially optimal distributions. Q.E.D.

Funding Note Open Access funding enabled and organized by Projekt DEAL.

References

Azmat, Ghazala and Iriberri, Nagore (2010). "The importance of relative performance feedback information: Evidence from a natural experiment using high school students." *Journal of Public Economics* 94(7-8): 435-452.

Bandiera, Oriana, Barankay, Iwan, and Rasul, Imran (2010). "Social incentives in the workplace." *Review of Economic Studies* 77(2): 417-458.

Bursztyn, Leonardo and Jensen, Robert (2015). "How does peer pressure affect educational investments?" *Quarterly Journal of Economics* 130(3): 1329-1367.

Cohn, Alain, Fehr, Ernst, Herrmann, Benedikt, and Schneider, Frédéric (2014). "Social comparison and effort provision: Evidence from a field experiment." *Journal of the European Economic Association* 12(4): 877-898.

Dobrescu, Loretti I., Faravelli, Marco, Megalokonomou, Rigissa, and Motta, Alberto (2021). "Relative performance feedback in education: Evidence from a randomised controlled trial." *Economic Journal* 131(640): 3145-3181.

Falk, Armin and Ichino, Andrea (2006). "Clean evidence of peer effects." *Journal of Labor Economics* 24(1): 39-57.

Garlick, Robert (2018). "Academic peer effects with different group assignment policies: Residential tracking versus random assignment." *American Economic Journal: Applied Economics* 10(3): 345-369.

Hanushek, Eric, Kain, John, Markman, Jacob, and Rivkin, Steven (2003). “Does peer ability affect student achievement?” *Journal of Applied Econometrics* 18(5): 527-544.
Heffetz, Ori and Frank, Robert H. (2011). “Preferences for status: Evidence and economic implications.” In Benhabib, Jess, Bisin, Alberto, and Jackson, Matthew (Eds.), *Handbook of Social Economics* Vol. 1A. Amsterdam: North-Holland, pp. 69-91.
Kafle, Kashi, Benfica, Rui, and Winters, Paul (2020). “Does relative deprivation induce migration? Evidence from Sub-Saharan Africa.” *American Journal of Agricultural Economics* 102: 999-1019.
Mas, Alexandre and Moretti, Enrico (2009). “Peers at work.” *American Economic Review* 99(1): 112-145.
Noy, Shakked and Sin, Isabelle (2021). “The effects of neighbourhood and workplace income comparisons on subjective wellbeing.” *Journal of Economic Behavior & Organization* 185: 918-945.
Sacerdote, Bruce (2001). “Peer effects with random assignment: Results for Dartmouth roommates.” *Quarterly Journal of Economics* 116(2): 681-704.
Shukla, Vibhooti and Stark, Oded (1986). “Urban external economies and optimal migration.” In Stark, Oded (Ed.), *Migration, Human Capital and Development*. Greenwich, CT: JAI Press, pp. 139-146.
Smith, Adam (1759). *The Theory of Moral Sentiments*. London: A. Millar.
Stark, Oded (1993). *The Migration of Labor*. Oxford and Cambridge, MA: Blackwell.
Stark, Oded (2017). “Migration when social preferences are ordinal: Steady-state population distribution and social welfare.” *Economica* 84(336): 647-666.
Stark, Oded and Taylor, J. Edward (1991). “Migration incentives, migration types: The role of relative deprivation.” *Economic Journal* 101(408): 1163-1178.